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The modified jump problem for the Laplace equation and singularities at the tips

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Abstract

The boundary value problem for the Laplace equation outside several cuts in a plane is studied. The jump of the solution of the Laplace equation and the boundary condition containing the jump of its normal derivative are specified of the cuts. The unique solution of this problem is obtained. The problem is reduced to the uniquely solvable Fredholm equation of the second kind and index zero. The singularities at the ends of the cuts are investigated.

Keywords: Harmonic functions; Cracks; Integral equations; Singularities at the edges

1. Introduction

Dirichlet and Neumann problems outside cuts in a plane for Laplace and Helmholtz equations were treated in [1–7,9–11,13]. The jump problem for the Laplace equation outside cuts in a plane has been studied in [8]. Two boundary conditions of jump type, namely, the jump of the unknown function and the jump of its normal derivative were specified at the cuts in [8]. In the present paper we extend our previous results [8] to the case of more general boundary condition. Unlike [8], the jump of the normal derivative of the unknown function is not given exactly, but expressed through limit values of this function on the cuts. As a result, uniqueness and solvability theorems obtained in the present paper are completely different from those in [8]. For example, if boundedness at infinity is assumed for a solution, then the present problem is uniquely solvable, while existence of a solution in [8] was subject to the necessary

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condition of solvability. Moreover, if this condition holds, then infinitely many solutions exist in [8]. Since boundary conditions in [8] were simpler than in the present problem, we succeeded in finding explicit representation for a solution there. The problem in the present paper is reduced to the uniquely solvable Fredholm integral equation without finding explicit representation for a solution, because the boundary conditions are more complicated. In the present paper we also give explicit formulas for singularities of the solution gradient at the ends of cuts. It appears that these singularities are weaker than in the Dirichlet and Neumann problems outside cuts in a plane [3,4]. The reason is such that the boundedness of jumps of unknown function and its normal derivative at the tips of cuts is a priori assumed in the jump problems. The properties of singularities of a solution gradient at the tips of cuts may be effectively used to select adequate model describing cracked media. Jump problems seems can be considered for modelling physical processes in cracked media or media with membranes as alternative to Dirichlet and Neumann problems.

2. Formulation of the problem

By a simple open curve we mean a non-closed smooth arc of finite length without self-intersections [11]. In the plane $x=(x_1,x_2)\in R^2$ we consider simple open curves $\Gamma_1,\ldots,\Gamma_N\in C^{2,\lambda}$, $\lambda\in(0,1]$, so that they do not have common points. We put $\Gamma=\bigcup_{n=1}^{N}\Gamma_n$. We assume that each curve Γ_n is parameterized by the are length s

$$\Gamma_n=\{x: x=x(s)=(x_1(s),x_2(s)),\ s\in[a_n,b_n]\},\quad n=1,\ldots,N,$$

so that $a_1<b_1<\cdots<a_N<b_N$. Therefore, points $x\in\Gamma$ and values of the parameter s are in one-to-one correspondence. Below the set of the intervals on the Os-axis $\bigcup_{n=1}^{N}[a_n,b_n]$ will be denoted by Γ also.

The tangent vector to Γ at the point $x(s)$ we denote by $\tau_x=(\cos\alpha(s),\sin\alpha(s))$, where $\cos\alpha(s)=x_1'(s)$, $\sin\alpha(s)=x_2'(s)$. Let $\mathbf{n}_x=(\sin\alpha(s),-\cos\alpha(s))$ be a normal vector to Γ at $x(s)$. The direction of $\mathbf{n}_x$ is chosen such that it will coincide with the direction of τ_x if $\mathbf{n}_x$ is rotated anticlockwise through an angle of $\pi/2$. We consider Γ as a set of cuts. The side of Γ which is on the left when the parameter s increases will be denoted by Γ^+ and the opposite side will be denoted by Γ^-.

We say that the function $u(x)$ belongs to the smoothness class $\mathbf{K}$ if the following conditions are satisfied:

(1) $u(x)\in C^0(\overline{R^2\backslash\Gamma})\cap C^2(R^2\backslash\Gamma)$ and $u(x)$ is continuous at the ends of Γ;

(2) $\nabla u\in C^0(\overline{R^2\backslash\Gamma}\backslash X)$, where X is a point set, consisting of the endpoints of Γ : $X=\bigcup_{n=1}^{N}(x(a_n)\cup x(b_n))$;

(3) in the neighbourhood of any point $x(d)\in X$, for some constants $\mathscr{C}>0$ and $\varepsilon>-1$, the inequality

$$|\nabla u|<\mathscr{C}|x-x(d)|^{\varepsilon} \tag{1}$$

holds, where $x\to x(d)$ and $d=a_n$ or $d=b_n$ for $n=1,\ldots,N$.

Remark. In the definition of the class $\mathbf{K}$ we consider Γ as a set of cuts in a plane. In particular, the notation $C^0(\overline{R^2\backslash\Gamma})$ denotes a class of functions, which are continuously extended on Γ from the left and right, but their values on Γ from the left and right can be different, so that the functions may have a jump across Γ.

Let us formulate the jump problem for the harmonic functions in $R^2\backslash\Gamma$.

Problem U. To find a function $u(x)$ of class **K**, so that $u(x)$ obeys the Laplace equation in $R^2\backslash\Gamma$

$$\triangle u = 0, \quad \triangle = \partial_{x_1}^2 + \partial_{x_2}^2 \tag{2}$$

satisfies the jump boundary conditions

$$u(x)|_{x(s)\in\Gamma^+} - u(x)|_{x(s)\in\Gamma^-} = f_1(s), \tag{3a}$$

$$\begin{gathered}\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{x(s)\in\Gamma^+} - \left.\frac{\partial u}{\partial \mathbf{n}}\right|_{x(s)\in\Gamma^-} = \beta(s)u(x)|_{x(s)\in\Gamma^+} + f_2(s),\\ \beta(s)\in C^{0,\lambda}(\Gamma), \quad \lambda\in(0,1]; \quad \beta(s)|_{s\in\Gamma}\not\equiv 0, \quad \beta(s)|_{s\in\Gamma}\leqslant 0\end{gathered} \tag{3b}$$

and meets the following conditions at infinity

$$|u(x)| < \mathscr{C}_1, \quad |\nabla u(x)| < \mathscr{C}_2|x|^{-2}. \tag{4}$$

where $\mathscr{C}_1, \mathscr{C}_2$ are some constants and $|x| = \sqrt{x_1^2 + x_2^2} \to \infty$. Functions $f_1(s),\ f_2(s)$ are given on Γ.

All conditions of the problem must be fulfilled in a classical sense. The case $\beta(s)\equiv 0$ has been studied in [8]. Gradient inequality in (4) is not necessary and follows from estimates for harmonic functions. We write this inequality for convenience only.

Conditions (1) at the ends of Γ in the formulation of the class **K** ensure the absence of point sources at the ends of Γ. If $f_1(s) = f_2(s) = 0$ on $\gamma\subset\Gamma$, then Eq. (2) holds on γ and $u(x)$ is analytic on γ.

Remark. Instead of the boundary condition (3b) we may consider another boundary condition

$$\begin{gathered}\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{x(s)\in\Gamma^+} - \left.\frac{\partial u}{\partial \mathbf{n}}\right|_{x(s)\in\Gamma^-} = \beta(s)u(x)|_{x(s)\in\Gamma^-} + f_0(s),\\ \beta(s)\in C^{0,\lambda}(\Gamma), \quad \lambda\in(0,1]; \quad \beta(s)|_{s\in\Gamma}\not\equiv 0, \quad \beta(s)|_{s\in\Gamma}\leqslant 0.\end{gathered} \tag{5}$$

However, this boundary condition can be easily reduced to (3b). Indeed, we substitute $u(x)|_{x(s)\in\Gamma^-}$ from (3a) to (5), then we arrive at (3b), where $f_2(s) = f_0(s) - \beta(s)f_1(s)$.

Theorem 1. *Problem* **U** *has at most one solution.*

By $\int_\Gamma \ldots \mathrm{d}\sigma$ we mean

$$\sum_{n=1}^{N}\int_{a_n}^{b_n} \ldots \mathrm{d}\sigma.$$

Now we prove the theorem. The limit values of functions on Γ^+ and Γ^- will be denoted by the superscripts "+" and "−", respectively.

Let $u_0(x)$ be a solution of the homogeneous problem **U**. We shall prove that $u_0(x) \equiv 0$. To prove this with the help of energy equalities for harmonic functions, we envelope open curves by closed contours, tend contours to the curves and use the smoothness of the solution of the problem **U**. In this way we obtain

$$\begin{aligned}\|\nabla u_0\|^2_{L_2(C_r\setminus\Gamma)} &= \int_\Gamma u_0^+\left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^+ \mathrm{d}s - \int_\Gamma u_0^-\left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^- \mathrm{d}s + \int_0^{2\pi} u_0 \frac{\partial u_0}{\partial r} r\,\mathrm{d}\varphi \\ &= \int_\Gamma \left\{(u_0^+ - u_0^-)\left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^- + u_0^+\left[\left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^+ - \left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^-\right]\right\} \mathrm{d}s + \int_0^{2\pi} u_0 \frac{\partial u_0}{\partial r} r\,\mathrm{d}\varphi,\end{aligned}$$

where C_r is the circle of the large radius r with the centre in the origin, and φ is a polar angle. We suppose, that $\Gamma \subset C_r$.

Since $u_0(x)$ meets zero boundary conditions of the homogeneous problem **U**, we get

$$\|\nabla u_0\|^2_{L_2(C_r\setminus\Gamma)} = -\int_\Gamma |\beta(s)|\,|u_0^+(x(s))|^2\,\mathrm{d}s + \int_0^{2\pi} u_0 \frac{\partial u_0}{\partial r} r\,\mathrm{d}\varphi.$$

Here, we took into account that $\beta(s) \leqslant 0$ for any $s \in \Gamma$. Putting $r \to \infty$, we have

$$\|\nabla u_0\|^2_{L_2(R^2\setminus\Gamma)} = \lim_{r\to\infty} \|\nabla u_0\|^2_{L_2(C_r\setminus\Gamma)} = -\int_\Gamma |\beta(s)|\,|u_0^+(x(s))|^2\,\mathrm{d}s \tag{6}$$

because

$$\lim_{r\to\infty} \int_0^{2\pi} u_0 \frac{\partial u_0}{\partial r} r\,\mathrm{d}\varphi = 0$$

according to conditions at infinity (4). It follows from (6) that $\|\nabla u_0\|^2_{L_2(R^2\setminus\Gamma)} = 0$ and, therefore $u_0(x) \equiv$ const. Since $\beta(s)$ is continuous on Γ and $\beta(s) \not\equiv 0$, there exists $s_0 \in \Gamma$, so that $\beta(s_0) \neq 0$. Hence, $\beta(s) \neq 0$ in the neighbourhood of s_0 on Γ. Owing to (6), $u_0^+(x(s_0)) = 0$. Consequently, const $= 0$ and $u_0(x) \equiv 0$ in R^2 (we used the smoothness of the function $u_0(x)$ ensured by the class **K**). Now the statement of the theorem follows from the linearity of the problem **U**.

3. The solution of the problem

To construct a solution of the problem **U** we impose additional assumptions on the functions $f_1(s)$, $f_2(s)$ in the boundary conditions (3)

$$f_1(s) \in C^{1,\lambda}(\Gamma), \quad f_2(s) \in C^{0,\lambda}(\Gamma), \quad \lambda \in (0, 1]; \tag{7a}$$

$$f_1(a_n) = f_1(b_n) = 0, \quad n = 1, \ldots, N. \tag{7b}$$

The solution of the problem **U** can be constructed in the form of a sum of a single layer potential, an angular potential [2,3] and a constant. Set

$$f_1'(\sigma) = \frac{\mathrm{d}}{\mathrm{d}\sigma} f_1(\sigma)$$

and consider a function

$$u[\mu](x) = v[f_1'](x) + w[\mu](x) + c[\mu] + c_0, \tag{8}$$

where

$$w[\mu](x) = -\frac{1}{2\pi}\int_\Gamma \mu(\sigma)\ln|x-y(\sigma)|\,\mathrm{d}\sigma$$

is a single layer potential for Eq. (2), and

$$v[f_1'](x) = -\frac{1}{2\pi}\int_\Gamma f_1'(\sigma)V(x,\sigma)\,\mathrm{d}\sigma$$

is the angular potential [2,3] for Eq. (2),

$$c[\mu] = \left\{\frac{1}{2\pi}\int_\Gamma \mu(\sigma)\int_\Gamma \beta(s)\ln|x(s)-y(\sigma)|\,\mathrm{d}s\,\mathrm{d}\sigma\right\}\Big/\int_\Gamma \beta(s)\,\mathrm{d}s,$$

$$c_0 = -\left\{\int_\Gamma\left(\beta(s)\left[\frac{1}{2}f_1(s)+\frac{1}{2\pi}\int_\Gamma f_1(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\ln|x(s)-y(\sigma)|\mathrm{d}\sigma\right]+f_2(s)\right)\mathrm{d}s\right\}\Big/\int_\Gamma \beta(s)\,\mathrm{d}s, \tag{9}$$

are constants. The integral term in the square brackets is the direct value of a double layer potential on Γ. The function $f_1(s)$ is specified in (3a) and

$$\int_\Gamma \beta(s)\,\mathrm{d}s \neq 0$$

under conditions of problem **U**. We will look for the unknown density $\mu(s)$ in the Hölder space $C^{0,\omega}(\Gamma)$ with some $\omega \in (0,1]$. The kernel $V(x,\sigma)$ is defined (up to indeterminacy $2\pi m$, $m=\pm1,\pm2,\ldots$) by the formulae

$$\cos V(x,\sigma) = \frac{x_1-y_1(\sigma)}{|x-y(\sigma)|}, \quad \sin V(x,\sigma) = \frac{x_2-y_2(\sigma)}{|x-y(\sigma)|},$$

where

$$y(\sigma) = (y_1(\sigma), y_2(\sigma)) \in \Gamma, \quad |x-y(\sigma)| = \sqrt{(x_1-y_1(\sigma))^2+(x_2-y_2(\sigma))^2}.$$

One can see, that $V(x,\sigma)$ is the angle between the vector $\overrightarrow{y(\sigma)x}$ and the direction of the Ox_1 axis. More precisely, $V(x,\sigma)$ is a many-valued harmonic function of x connected with $\ln|x-y(\sigma)|$ by the Cauchy–Riemann relations. Below by $V(x,\sigma)$ we denote an arbitrary fixed branch of this function, which varies continuously with σ along each curve Γ_n $(n=1,\ldots,N)$ for given fixed $x \notin \Gamma$. Under this definition of $V(x,\sigma)$, the potential $v[f_1'](x)$ is a many-valued function. In order that the potential $v[f_1'](x)$ be single-valued, the following additional conditions [2] must hold:

$$\int_{a_n}^{b_n} f_1'(\sigma)\,\mathrm{d}\sigma = f_1(b_n) - f(a_n) = 0, \quad n=1,\ldots,N.$$

Clearly, these conditions are satisfied due to our assumptions (7b). Integrating $v[f_1'](x)$ by parts and using (7b) we express the angular potential in terms of a double layer potential

$$v[f_1'](x) = \frac{1}{2\pi}\int_\Gamma f_1(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\ln|x-y(\sigma)|\,\mathrm{d}\sigma. \tag{10}$$

Consequently, the angular potential $v[f_1'](x)$ satisfies Eq. (2) outside Γ and conditions at infinity (4). The single layer potential $w[\mu](x)$ meets conditions (4) if

$$\int_\Gamma \mu(s)\mathrm{d}s = 0. \tag{11}$$

It follows from properties of single layer and angular potentials [3,12] that the function (8) belongs to the class **K** and satisfies Eq. (2). Besides, function (8) meets conditions (4) at infinity if (11) holds. Hence, if (11) holds, then the function (8) satisfies all conditions of problem **U** except the boundary conditions (3). To satisfy the boundary conditions we, at first, derive the jump formulas for $u(x)$ and its normal derivative on Γ using limit properties of single layer potential and angular potential. According to [3], normal derivative of the angular potential $v[f_1'](x)$ is continuous across Γ. The single layer potential $w[\mu](x)$ is continuous across Γ in our assumptions. On the basis of the jump relations on Γ for the angular potential and for the normal derivative of the single layer potential, we obtain [3]

$$\begin{aligned} u[\mu](x)|_{x(s)\in\Gamma_n^+} - u[\mu](x)|_{x(s)\in\Gamma_n^-} &= v[f_1'](x)\big|_{x(s)\in\Gamma_n^+} - v[f_1'](x)\big|_{x(s)\in\Gamma_n^-} \\ &= \int_{a_n}^{s}\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}f_1(\sigma)\right)\mathrm{d}\sigma = f_1(s), \quad n=1,\ldots,N, \end{aligned} \tag{12}$$

$$\frac{\partial u[\mu](x)}{\partial \mathbf{n}_x}\bigg|_{x(s)\in\Gamma^+} - \frac{\partial u[\mu](x)}{\partial \mathbf{n}_x}\bigg|_{x(s)\in\Gamma^-} = \frac{\partial}{\partial \mathbf{n}_x}w[\mu](x)\bigg|_{x(s)\in\Gamma^+} - \frac{\partial}{\partial \mathbf{n}_x}w[\mu](x)\bigg|_{x(s)\in\Gamma^-} = \mu(s), \tag{13}$$

where conditions (7b) for $f_1(s)$ have been employed. It follows from (12) that the function (8) satisfies the boundary condition (3a). Substituting (8) in (3b), using (13) and limit formulas for potentials [3], we obtain the integral equation for the function $\mu(s)$ on Γ

$$\mu(s) + \beta(s)\frac{1}{2\pi}\int_\Gamma \mu(\sigma)\ln|x(s)-y(\sigma)|\,\mathrm{d}\sigma - \beta(s)c[\mu] = F(s), \quad s\in\Gamma, \tag{14}$$

where

$$F(s) = f_2(s) + \beta(s)\left[\frac{1}{2}f_1(s) + \frac{1}{2\pi}\int_\Gamma f_1(\sigma)\frac{\partial \ln|x(s)-y(\sigma)|}{\partial \mathbf{n}_y}\,\mathrm{d}\sigma\right] + \beta(s)c_0 \tag{15}$$

and conditions (7b) have been used. The constants $c[\mu]$, c_0 were introduced in (9). The term in the square brackets is the limit value of a double layer potential (10) on Γ^+. The last term in square brackets is the direct value of the double layer potential (10) on Γ. This term belongs to $C^{0,\lambda}(\Gamma)$ in s (see [4, Lemma 1] and [3, Lemma 2]). Taking into account conditions (7a) we observe that $F(s)\in C^{0,\lambda}(\Gamma)$. The kernel of the integral term in (14) has logarithmic singularity if $s=\sigma$ and can be represented in the form

$$\ln|x(s)-y(\sigma)| = \ln|s-\sigma| + g_0(s,\sigma) = \frac{g(s,\sigma)}{|s-\sigma|^{1/4}}, \tag{16}$$

where $g_0(s,\sigma) \in C^1(\Gamma)$, $g(s,\sigma) = |s-\sigma|^{1/4} \ln|x(s) - y(\sigma)| \in C^{0,\omega_0}(\Gamma)$ for any $\omega_0 \in (0, 1/4)$. The integral operator in (14)

$$\mathbf{A}\mu = \int_\Gamma \mu(\sigma) \ln|x(s) - y(\sigma)|\,\mathrm{d}\sigma$$

is a compact operator mapping $C^0(\Gamma)$ into itself, because it has a weak singularity in the kernel [12]. Moreover, one can check by direct calculations using (16) that this integral operator maps $C^0(\Gamma)$ into $C^{0,\omega_0}(\Gamma)$ for any $\omega_0 \in (0, \frac{1}{4})$. Hence if $\mu(s)$ is a solution of (14) in $C^0(\Gamma)$, then proceeding from identity (14) for $\mu(s)$ we observe that $\mu(s)$ automatically belongs to $C^{0,\omega}(\Gamma)$, where $\omega \in (0, \min\{\lambda, 1/4\})$. It can be checked directly that any solution of Eq. (14) in $C^0(\Gamma)$ automatically satisfies condition (11), therefore function (8) with such density meets conditions (4) at infinity. Consequently, if $\mu(s) \in C^0(\Gamma)$ is a solution of Eq. (14), then according to properties of potentials [2,3,12] the function (8) belongs to the class **K** and satisfies all conditions of the problem **U**.

We arrive at the lemma.

Lemma. 1. *Let $F(s) \in C^0(\Gamma)$. Then any solution of Eq.* (14) *in $C^0(\Gamma)$ meets condition* (11).

2. *If $F(s) \in C^{0,\lambda}(\Gamma)$, $\lambda \in (0, 1]$, then any solution of Eq.* (14) *in $C^0(\Gamma)$ automatically belongs to $C^{0,\omega}(\Gamma)$ with any $\omega \in (0, \min\{\lambda, 1/4\})$.*

3. *Let conditions* (7) *hold. If $\mu(s)$ is a solution of Eq.* (14) *in $C^0(\Gamma)$, where $F(s)$ is given by* (15), *then function* (8) *is a solution of the problem* **U**.

Below we look for a solution of (14) in $C^0(\Gamma)$. Eq. (14) in $C^0(\Gamma)$ is a Fredholm integral equation of the second kind and index zero, since its kernel has a weak singularity as aforementioned.

Let us show that the homogeneous equation (14) has only a trivial solution in $C^0(\Gamma)$. Let $\mu^0(s)$ be a solution of the homogeneous equation (14). According to the Lemma, $\mu^0(s)$ automatically belongs to $C^{0,\omega}(\Gamma)$. It follows from point 3 of the lemma that

$$u[\mu^0](x) = w[\mu^0](x) + c[\mu^0] + c_0 = \frac{1}{2\pi}\int_\Gamma \mu^0(s) \ln|x(s) - y(\sigma)|\,\mathrm{d}\sigma + c[\mu^0] + c_0$$

is a solution of the homogeneous problem **U**. Taking into account Theorem 1, we observe that $u[\mu^0](x) \equiv 0$, $x \in R^2\backslash\Gamma$, since homogeneous Problem **U** has only a trivial solution. Using the jump relations for the normal derivative of the single layer potential on Γ we obtain

$$\left.\frac{\partial u[\mu^0](x)}{\partial \mathbf{n}_x}\right|_{x\in\Gamma^+} - \left.\frac{\partial u[\mu^0](x)}{\partial \mathbf{n}_x}\right|_{x\in\Gamma^-} = \mu^0(s) \equiv 0, \quad s \in \Gamma.$$

Thus, we have proved that the homogeneous equation (14) has only a trivial solution. According to Fredholm alternative, the inhomogeneous equation (14) is uniquely solvable in $C^0(\Gamma)$ for any $F(s) \in C^0(\Gamma)$.

Theorem 2. *Eq.* (14) *has unique solution $\mu(s) \in C^0(\Gamma)$ for any $F(s) \in C^0(\Gamma)$. In addition, if $F(s) \in C^{0,\lambda}(\Gamma)$, $\lambda \in (0, 1]$, then the unique solution of Eq.* (14) *in $C^0(\Gamma)$ belongs to $C^{0,\omega}(\Gamma)$ with $\omega \in (0, \min\{\lambda, 1/4\})$.*

Note that if conditions (7) hold, then $F(s) \in C^{0,\lambda}(\Gamma)$, $\lambda \in (0,1]$. Using the lemma we obtain solvability of the Problem **U**.

Theorem 3. *If conditions* (7) *hold, then the solution of the Problem* **U** *exists and is given by* (8), *where* $\mu(s) \in C^{0,\omega}(\Gamma)$ $(\omega \in (0, \min\{\lambda, 1/4\}))$ *is a solution of Eq.* (14) *ensured by Theorem* 2.

It can be verified by direct calculations that condition (1) for $|\nabla u|$ is fulfilled for any $\varepsilon \in (0,1)$, i.e. for any small positive ε. In other words, $\nabla u(x)$ does not have power singularity at the ends of Γ. It will be shown in the next section that ∇u has logarithmic singularity or, in certain cases, does not have singularity at all. Explicit formulas for singularities of ∇u at the ends of Γ will be presented and discussed in the next section.

4. Singularities of a gradient of a solution at the ends of Γ

In this section by $u(x)$ we denote the solution of Problem **U** ensured by Theorem 3. According to (1), ∇u may be unbounded at the ends of Γ. The explicit expressions for singularities of ∇u can be obtained from the formulas for singularities of derivatives of single layer and angular potentials near edges [3,4]. Let $x(d)$ be one of the end-points of Γ. In the neighbourhood of $x(d)$ we introduce the system of polar coordinates $x_1 = |x - x(d)| \cos\varphi$, $\quad x_2 = |x - x(d)| \sin\varphi$. We will assume that $\varphi \in (\alpha(d), \alpha(d) + 2\pi)$ if $d = a_n$ and $\varphi \in (\alpha(d) - \pi, \alpha(d) + \pi)$ if $d = b_n$ $(n = 1, \dots, N)$. Recall that $\alpha(s)$ is the angle between the tangent vector τ_x to Γ at the point $x(s)$ and the direction of the Ox_1-axis. Hence, $\alpha(d) = \alpha(a_n + 0)$ if $d = a_n$ and $\alpha(d) = \alpha(b_n - 0)$ if $d = b_n$. Consequently, the angle φ varies continuously in the neighbourhood of the point $x(d)$, cut along the contour Γ.

Recall that X is a set of end-points of Γ. Computing singularities of ∇u in the same way as in [3,4] we arrive at the following assertion.

Theorem 4. *Let* $x \to x(d) \in X$. *Then in the neigbourhood of the point* $x(d)$ *the derivatives of the solution of Problem* **U** *have the following behaviour*:

$$\begin{aligned}\frac{\partial}{\partial x_1} u(x) = & -(-1)^m \frac{f_1'(d)}{2\pi}[-\sin\alpha(d) \ln|x - x(d)| + \varphi \cos\alpha(d)] \\ & -(-1)^m \frac{\mu(d)}{2\pi}[\cos\alpha(d) \ln|x - x(d)| + \varphi \sin\alpha(d)] + \mathrm{O}(1),\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x_2} u(x) = & -(-1)^m \frac{f_1'(d)}{2\pi}[\cos\alpha(d) \ln|x - x(d)| + \varphi \sin\alpha(d)] \\ & +(-1)^m \frac{\mu(d)}{2\pi}[-\sin\alpha(d) \ln|x - x(d)| + \varphi \cos\alpha(d)] + \mathrm{O}(1),\end{aligned}$$

where $m = 0$ *if* $d = a_n$ *and* $m = 1$ *if* $d = b_n$ $(n = 1, \dots, N)$.

Remark. By $\mathrm{O}(1)$ we denote functions which are continuous at the point $x(d)$. Furthermore, the functions denoted by $\mathrm{O}(1)$ are continuous in the neighbourhood of the point $x(d)$, cut along the contour Γ.

According to Theorem 4, ∇u has logarithmic singularities at the ends of cuts Γ in general. However, if $f_1'(d) = \mu(d) = 0$ at the end $x(d) \in X$, then there is no singularity of ∇u at the end $x(d)$. Moreover, ∇u is continuous at this end. If $f_1'(d) \neq 0$ or $\mu(d) \neq 0$, then ∇u has a logarithmic singularity at $x(d) \in X$.

Let us compare our results with singularities of a solution gradient in the Dirichlet and Neumann problems at the exterior of cuts in a plane. In these problems either Dirichlet or Neumann boundary condition has been specified on the cuts instead of (3). It was shown in [3,4] that the solution gradient in the Dirichlet and Neumann problems in general tends at infinity as $O(|x - x(d)|^{-1/2})$ when $x \to x(d) \in X$. According to Theorem 3, the edge singularities of ∇u in the jump problem are generally logarithmic. Thus, the jump problem and Dirichlet/Neumann problem have as a rule different orders of singularities at the ends of cuts, so that the singularities in the jump problem are weaker. We can conclude that the behaviour of the solution in the jump problem is essentially different from behaviour of the solution in the Dirichlet/Neumann problem. The discussed properties of singularities may be effectively used to select adequate model describing wave propagation cracked in media or media with membranes.

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